Int. J. Heat Mass Transfer. Vol. 13, pp. 413–427. Pergamon Press 1970. Printed in Great Britain

AXISYMMETRIC MELTING OR SOLIDIFICATION OF CIRCULAR CYLINDERS*

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(*Received* 3 *June* 1968 *and in revised form* 10 *April* 1969)

Abstract—Analytical short-time solution and numerical solutions for all times are derived for a cylinder melting under arbitrary heat inputs applied on its outer radius. The solutions are obtained by the embedding technique; the accuracy of the solutions is discussed and some numerical results are presented.

NOMENCLATURE

$a(t)$, depth of ablated material;
b_0, inner radius;
b, outer radius;
c, specific heat;
c_i, coefficient of fictitious heat flux expansion defined in equation (3.7a);
D, κ_L/κ_S;
d_i, coefficient of melt depth expansion defined in equation (3.7b);
k, conductivity;
l, latent heat;
m, non-dimensional material property defined in equation (3.4);
Q, heat flux;
Q_0, reference heat flux defined in equation (3.4);
q_i, coefficient in applied heat flux expansion;
R, non-dimensional quantity related to reference heat flux defined in equation (3.4);
r, spatial coordinate;
$s(t)$, depth of melted material;
T, temperature;
T_m, melt temperature of material;
$T_I(r)$, initial temperature distribution;
$T_I^*(r)$, $T_I^*(X)$, initial temperature distribution in fictitious liquid;
$T_I^*(r, t)$, $T_I^*(X, y)$, temperature distribution in fictitious liquid;
$\bar{T}_0(r, t)$, $T_0(X, y)$, Green's function;
t, time;
t_m, time for melt initiation;
t_f, time for melt completion;
X, $(r - b_0)/(b - b_0)$;
y, $(t - t_m)/t_m$;
y_m, $\kappa t_m/(b - b_0)^2$;
y_f, $\kappa t_f/(b - b_0)^2$;
y^*, limiting time for applicability of short-time solution;
δ, time interval;
κ, diffusivity;
$\xi(y)$, non-dimensional melt depth defined in equation (3.4);
ρ, density.

Subscripts

S, variable associated with solid;
L, variable associated with liquid.

* This work is based on a dissertation submitted in partial fulfilment of the requirements for the degree of Doctor of Engineering Science in the School of Engineering and Applied Science, Columbia University, and was partially supported by the Office of Naval Research.

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Superscripts

′, variable associated with fictitious body;

*, variable associated with analytic continuation of pre-melt function.

1. INTRODUCTION

THE PROBLEM of heat conduction in a body with a moving interface at which energy is absorbed or emitted is currently of considerable interest. While it has always been of concern to metal fabricators, it is only recently, particularly with the use of ablating heat shields, that it has become the subject of intense study. The problem is non-linear and is further complicated by the facts that the thermal properties in the liquid and solid phases are different and that the thermal and flow problems are interdependent.

Numerical solutions of the axisymmetric melting of a cylinder have been obtained, for example, by Lardner and Pohle [1] by means of the heat balance integral, by Bonilla and Strupezewski [2] with the aid of an analog computer, and by Springer and Olson [3] by means of a finite difference space-and-time grid. In the present work are included short-time analytical solutions, as well as numerical long-time solutions which are obtained by a forward integration in the time domain only.

The principal purposes of this investigation are the construction of analytical short-time solutions of the above-mentioned problem and the assessment of the feasibility of using the "embedding technique" for obtaining long-time numerical solutions. In the course of the work some numerical results of practical interest are derived.

In Section 2 the general problem of ablation of an axisymmetric circular cylinder, solid or hollow, is mathematically defined and in Section 3 the problem is solved for the case of instantaneous melt removal. The method of solution, called the embedding technique and first introduced by Boley [4], can be applied to materials with non-uniform but temperature-independent properties, and to problems with arbitrary melt removal and applied heat fluxes which are time and space dependent. The method requires the introduction of a fictitious body whose shape is unchanged and identical with that of the unmelted body throughout the heating process. An unknown fictitious heat flux is applied to the surface of this body and is adjusted so as to satisfy, at points on the boundary of the actual body, the boundary conditions of the real problem. This results in a form of the inverse problem of transient heat conduction, in which however the interior conditions are implicitly prescribed.

In Section 4 the results are discussed, and it is found that the adjustment of the unknown surface flux to give the desired conditions in the interior leads to certain numerical difficulties. Stolz [5] and Mirsepassi [6] found this generally to be true for the inverse problem. Sparrow *et al.* [7] consider the inverse problem and suggest a technique to avoid instabilities which makes use of a predictor–corrector method. In this paper numerical solutions of two distinct approaches are presented; in the first an extrapolation technique is employed while in the latter Sparrow's suggestion is adapted to the present problem.

In Section 5 a general discussion of the short-time results is presented and the solidification problem, or melting with no melt removal, is briefly considered.

2. AXISYMMETRIC MELTING OF A CIRCULAR CYLINDER WITH ABLATION

Consider a right circular cylinder (Fig. 1) insulated at the ends and at its inner radius, heated on its outer cylindrical surface by a prescribed flux $Q(r, t)$ and initially ($t = 0$) at a temperature $T_I(r) < T_m$, where T_m is the melting temperature of the material.* The temperature on the boundary of the cylinder ($r = b$) increases

* Only melting is discussed explicitly for the sake of concreteness, but the results may be applied equally as well to solidification since the equations take the same form in both cases when written using dimensionless notation.

as heating progresses and reaches, at a time $t_m > 0$, the value T_m. For later times, as heating continues, a portion of the cylinder melts, so that the remaining solid region has an outer radius $b - s(t)$. The melted portion may either be removed or left standing upon formation. The mathematical formulation of the problem differs in the pre-melt and post-melt regimes, and may be stated as follows, on the assumption that the properties are uniform and constant, but not necessarily equal in the solid and liquid regions:

Pre-melt problem ($t \leqslant t_m$):*

$$\frac{\partial^2 T_{0S}}{\partial r^2} + \frac{1}{r}\frac{\partial T_{0S}}{\partial r} = \frac{1}{\kappa_S}\frac{\partial T_{0S}}{\partial t} \qquad b_0 < r < b, \quad t > 0, \tag{2.1}$$

$$k_S \frac{\partial T_{0S}}{\partial r} = Q(r, t) \qquad r = b, \quad t > 0 \tag{2.2a}$$

$$T_{0S}(r, t) = T_I(r) \qquad b_0 < r < b, \quad t = 0, \tag{2.2b}$$

$$k_S \frac{\partial T_{0S}}{\partial r} = 0 \qquad r = b_0, \quad t > 0. \tag{2.2c}$$

Post-melt problem ($t > t_m$):

$$\frac{\partial^2 T_S}{\partial r^2} + \frac{1}{r}\frac{\partial T_S}{\partial r} = \frac{1}{\kappa_S}\frac{\partial T_S}{\partial t} \qquad b_0 < r < b - s(t), \tag{2.3}$$

$$\frac{\partial^2 T_L}{\partial r^2} + \frac{1}{r}\frac{\partial T_L}{\partial r} = \frac{1}{\kappa_L}\frac{\partial T_L}{\partial t} \qquad b - s(t) < r < b - a(t), \tag{2.4}$$

$$T_S(r, t) = T_m \qquad r = b - s(t), \tag{2.5a}$$

$$k_S \frac{\partial T_S}{\partial r} = Q_2(t) - \rho_S l_S \frac{\mathrm{d}s}{\mathrm{d}t} \qquad r = b - s(t), \tag{2.5b}$$

$$k_S \frac{\partial T_S}{\partial r} = 0 \qquad r = b_0, \tag{2.5c}$$

$$T_L(r, t) = T_m \qquad r = b - s(t), \tag{2.6a}$$

$$k_L \frac{\partial T_L}{\partial r} = Q_2(t) \qquad r = b - s(t), \tag{2.6b}$$

$$k_L \frac{\partial T_L}{\partial r} = Q(r, t) \qquad r = b - a(t), \tag{2.7}$$

$$T_S(r, t) = T_{0S}(r, t_m) \qquad b_0 < r < b, \quad t = t_m, \tag{2.8}$$

where $a(t)$ is a prescribed depth of ablated or removed liquid material. Equation (2.5b) represents the heat balance across the interface. The function $Q_2(t)$ in equations (2.5b) and (2.6) is the flux leaving the liquid at the solid–liquid interface and couples the solution for the solid and the liquid. In the special case in which the melt is instantaneously removed upon formation, equations (2.4), (2.6) and (2.7) are dropped and $Q(r, t)$ replaces $Q_2(t)$ in equation (2.5b). In this case the applied flux is then assumed to be

* The subscripts S and L are used to denote solid and liquid, respectively.

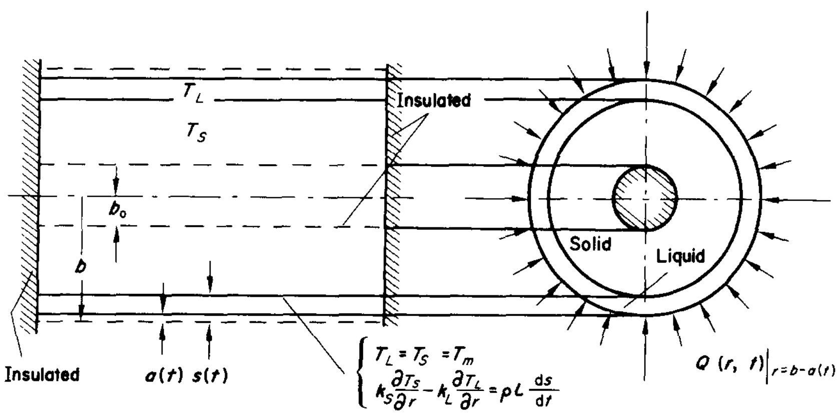

FIG. 1. Melting of an axisymmetrically heated circular cylinder.

positive since with instantaneous removal a nonpositive flux results in the cessation of melting, in which case the pre-melt formulation again applies.

It is not possible to obtain a general closed-form solution to this non-linear problem; hence we develop: (a) analytical solutions valid for short times and (b) extended numerical solutions for the special limiting case of instantaneous melt removal. The other limiting case of no melt removal is briefly discussed later, along with some qualitative results regarding the short-time solution.

Some conclusions which follow from the above formulation and are generally useful in later developments will first be given. The total heat balance for the case of arbitrary removal is

$$\begin{aligned}\int_0^{t_m} bQ(b,t)\,\mathrm{d}t &+ \int_{t_m}^{t} (b-a)\,Q(b-a,t)\,\mathrm{d}t \\ &= \int_{b-s}^{b} [\rho_S l_S + \rho_S c_S(T_m - T_I)]\,r\,\mathrm{d}r \\ &+ \int_{b_0}^{b-s} \rho_S c_S(T_S - T_I)\,r\,\mathrm{d}r \\ &+ \int_{b-s}^{b-a} \rho_L c_L(T_L - T_m)\,r\,\mathrm{d}r \qquad (2.9)\end{aligned}$$

where for the case of instantaneous melt removal $a(t) \equiv s(t)$ and the last integral on the right is zero. For this case, when Q is constant, the total time to melt is

$$t_f = \frac{1}{bQ}\int_{b_0}^{b} [\rho_S l_S + \rho_S c_S(T_m - T_I)]\,r\,\mathrm{d}r + \int_{t_m}^{t_f} \frac{s(t)}{b}\,\mathrm{d}t. \qquad (2.10)$$

This equation cannot be used to evaluate t_f until the solution $s(t)$ is known; however, as will be seen, it can provide a useful check on the numerical solutions which will be obtained later.

The initial and final melt rates for the case of instantaneous liquid removal can be written without determining the solution explicitly. By subtracting equation (2.2a) evaluated at $t = t_m^- - t_f$ from equation (2.5a) and noting that $\partial T_S/\partial r$ is continuous at $t = t_m$ one obtains

$$\rho_S l_S \frac{\mathrm{d}s}{\mathrm{d}t}(t_m^+) = Q(b, t_m^+) - Q(b, t_m^-). \qquad (2.11)$$

For the final melt rate, as melting is completed, one need only note that $\partial T_S/\partial r$ is always zero at $r = b_0$, so that, from equation (2.5b),

$$\rho_S l_S \frac{\mathrm{d}s}{\mathrm{d}t}(t_f^-) = Q(b_0, t_f^-). \qquad (2.12)$$

3. INSTANTANEOUS MELT REMOVAL

In the embedding technique for the case of instantaneous melt removal, i.e. $a(t) = s(t)$, the melting cylinder of unknown outer radius is replaced by a fictitious one of fixed radius b

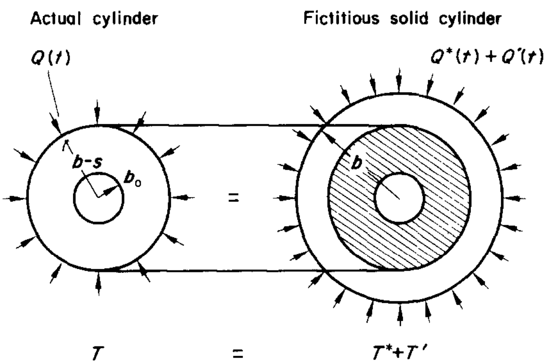

FIG. 2. Actual and fictitious cylinders for problem with instantaneous melt removal. Fictitious cylinder completely envelopes the actual cylinder and has a fixed outer radius b.

(Fig. 2). A total flux $Q^*(t) + Q'(t)$ is applied to the surface of this cylinder, where $Q^*(t)$ is the analytic continuation of the pre-melting heat flux and $Q'(t)$ is the unknown fictitious heat flux. Thus, the temperature distribution is the sum of the analytic continuation of the pre-melt temperature distribution, $T^*(r,t)$, and of an unknown distribution $T'(r,t)$ due to $Q'(t)$. With Duhamel's theorem,

$$T^*(r,t) = \int_0^t Q^*(t - t_1)\frac{\partial \bar{T}_0(r,t_1)}{\partial t_1}\,\mathrm{d}t_1, \qquad (3.1)$$

$$T'(r,t) = \int_0^{t-t_m} Q'(t-t_1)\frac{\partial \overline{T}_0(r,t_1)}{\partial t_1}\mathrm{d}t_1. \qquad (3.2)$$

Where $\overline{T}_0(r,t)$ is the temperature in the cylinder, initially at zero temperature, subjected to a unit flux on its outer surface and insulated at its inner surface. In this form the temperature already satisfies both the Fourier heat conduction equation, equation (2.3), and the initial temperature distribution in the actual melting body, equation (2.8). We now write the boundary conditions on the actual body at $r = b - s(t)$ in terms of the temperatures just defined, namely

$$T_m = T^*(r,t) + \int_0^{t-t_m} Q'(t-t_1)\frac{\partial \overline{T}_0}{\partial t_1}\mathrm{d}t_1$$

$$r = b - s(t), \qquad (3.3a)$$

$$Q(r,t) - \rho l\frac{\mathrm{d}s}{\mathrm{d}t} = k\frac{\partial T^*}{\partial r} + k\int_0^{t-t_m} Q'(t-t_1)\frac{\partial^2 \overline{T}_0}{\partial r\partial t_1}\mathrm{d}t_1$$

$$r = b - s(t). \qquad (3.3b)$$

This is a system of ordinary integro-differential equations to be solved for the two unknowns $Q'(t)$ and $s(t)$.

The following non-dimensional variables will be introduced:

$$y_1 = \frac{t_1}{t_m} \qquad y = \frac{t}{t_m} - 1 \qquad X = \frac{r - b_0}{b - b_0}$$

$$\xi(y) = \frac{b - b_0}{2\sqrt{(\kappa t_m)}}\frac{s(t)}{b - b_0} \qquad y_m = \frac{\kappa t_m}{(b - b_0)^2}$$

$$m = \frac{\sqrt{\pi}}{2}\frac{cT_m}{l} \qquad R = \frac{Q_0(b - b_0)}{kT_m}$$

$$Q_0 = \frac{\sqrt{\pi}}{2}\frac{kT_m}{\sqrt{(\kappa t_m)}}$$

$$T_0(\kappa, y) = \frac{k}{b - b_0}\overline{T}_0(r,t). \qquad (3.4)$$

Q_0 is a constant reference flux taken as the constant heat input which melts a semi-infinite body, with the same properties as the cylinder, in the time t_m. Equations (3.3) then become

$$\int_0^y \frac{Q'(y-y_1)}{Q_0}\frac{\partial T_0}{\partial y_1}\mathrm{d}y_1 = \frac{1}{R}\left(1 - \frac{T^*}{T_m}\right)$$

$$X = 1 - 2\sqrt{(y_m)}\,\xi(y) \qquad (3.5a)$$

$$\int_0^y \frac{Q'(y-y_1)}{Q_0}\frac{\partial^2 T_0}{\partial X\partial y_1}\mathrm{d}y_1 = \frac{Q(X,y)}{Q_0} - \frac{2}{m}\frac{\mathrm{d}\xi}{\mathrm{d}y} - \frac{1}{RT_m}\frac{\partial T^*}{\partial X}$$

$$X = 1 - 2\sqrt{(y_m)}\,\xi(y). \qquad (3.5b)$$

The function T_0 for solid cylinder with $b_0 = 0$ can be found in the literature and is usually given in two forms, the first convergent for all X and y,

$$T_0(X,y) = 2\left\{y_m y + \frac{1}{4}(x^2 - \tfrac{1}{2}) - \sum_{s=1}^{\infty}\exp(-\alpha_s^2 y_m y)\frac{J_0(\alpha_s X)}{\alpha_s J_0(\alpha_s)}\right\}, \qquad (3.6a)$$

with α_s determined from $J_1(\alpha) = 0$, and the second,

$$\begin{aligned} T_0(X,y) \sim \frac{1}{\sqrt{X}}\Bigg\{ & 2\sqrt{(y_m y)}\, i\,\mathrm{erfc}\frac{1-X}{2\sqrt{(y_m y)}} + \frac{y_m y}{X}\left(3 + \frac{1}{X}\right) i^2\,\mathrm{erfc}\frac{1-X}{2\sqrt{y_m y}} \\ & + \frac{3}{8}(y_m y)^{\frac{3}{2}}\left(\frac{11}{2.} + \frac{1}{X} + \frac{3}{2X^2}\right) \times i^3\,\mathrm{erfc}\frac{1-X}{2\sqrt{(y_m y)}} + \frac{3}{64}(y_m y)^2\Bigg(83 + \frac{11}{X} + \frac{9}{X^2} + \frac{25}{X^3}\Bigg) \\ & \times i^4\,\mathrm{erfc}\frac{1-X}{2\sqrt{(y_m y)}} + \ldots\Bigg\}, \end{aligned} \qquad (3.6b)$$

which is an asymptotic expansion valid for small values of y and of $(1 - X)$, that is in the region where the first representation converges very slowly. Note that for small values of y the function T_0 is the same for solid and hollow cylinders; hence the short-time results derived in this section are equally valid for both.

Equations (3.5) are valid for all times and for arbitrary heat fluxes. They are solved analytically in detail for short times after the start of melt in [13] and in this paper numerically for all times. Briefly, the short-time solution is obtained by expanding the known and unknown functions appearing in the equations about $y = 0$ and equating coefficients of like terms in y. One writes, in other words,

$$\frac{Q'(y)}{Q} = c_1 y^{\frac{1}{2}} + c_2 y + c_3 y^{\frac{3}{2}} + \ldots, \qquad (3.7a)$$

$$\xi(y) = d_2 y + d_3 y^{\frac{3}{2}} + d_4 y^2 + \ldots \qquad (3.7b)$$

Table 1. Coefficients of extended short time solution

Order of term	Instantaneous melt removal $\xi(y)$	Constant applied flux $Q^1(y)/Q_0$
$y^{\frac{1}{2}}$	0	$-\frac{4}{\pi} a_{11}$
y	0	$\frac{1}{\sqrt{\pi}}\left(\sqrt{y_m} - \frac{4}{\sqrt{\pi}} m\sqrt{y_m} a_{10}\right) a_{11}$
$y^{\frac{3}{2}}$	$\frac{4}{3\pi} m a_{11}$	$\frac{16}{3\pi}\left\{- a_{22} + \left[-\frac{2}{\pi} m^2 y_m a_{10}^2 - \frac{16}{3\pi^{312}} m a_{11} + -\frac{1}{\sqrt{\pi}} m y_m a_{10} + \frac{1}{16} y_m\right] a_{11}\right\}$
y^2	$\frac{1}{\sqrt{\pi}}\left(\sqrt{y_m} - \frac{4}{\sqrt{\pi}} m\sqrt{y_m} a_{10}\right) a_{11}$	$-\frac{4}{\pi} m\sqrt{y_m} a_{10} a_{22} + \frac{2}{\sqrt{\pi}}\sqrt{y_m} a_{22} + \left[-\frac{8}{\pi^2} m^3 y_m^{\frac{3}{2}} a_{10}^3 - \frac{8}{\pi^{\frac{3}{2}}} m^2 y_m^{\frac{3}{2}} a_{10} a_{20} - \frac{1}{\pi^{\frac{3}{2}}}\left(\frac{27}{2} + \frac{64}{3\pi}\right) m^2 \sqrt{y_m} a_{10} a_{11} + \frac{2}{\pi^{\frac{3}{2}}} m^2 y_m^{\frac{3}{2}} a_{10}^2 + \left(\frac{16}{3\pi} + \frac{35}{8}\right)\frac{1}{\pi} m\sqrt{y_m} a_{11} + \frac{2}{8\sqrt{\pi}} y_m^{\frac{3}{2}}\right] a_{11}$
y	$\frac{16}{15\pi} m\left\{a_{22} + \left[\frac{2}{\pi} m^2 y_m a_{10}^2 - \frac{1}{16} y_m \quad \frac{2}{\sqrt{\pi}} m y_m a_{20} + \left(\frac{16}{3\pi} - 1\right)\frac{1}{\sqrt{\pi}} m a_{11}\right] a_{11}\right\}$	
y^3	$-\frac{1}{6} m c_4 + \frac{1}{6\pi} m\left\{-\frac{4}{\sqrt{\pi}}\left(1 + \frac{16}{3\pi}\right) m^2 \sqrt{y_m} a_{10} a_{11} + \frac{8}{\sqrt{\pi}} m^2 y_m^{\frac{3}{2}} a_{10} \hat{a}_{20} - 2 m y_m^{\frac{3}{2}} a_{20} + 4\left(1 + \frac{4}{3\pi}\right) m\sqrt{y_m} a_{11}\right\} a_{11}$	

Table 2. Coefficients of extended short time solution

	Instantaneous melt removal	Variable applied flux
Order of term	$\xi(y)$	$Q'(y)/Q_0$
$y^{\frac{1}{2}}$	0	$-\frac{4}{\pi}(a_{11} - m\sqrt{y_m}a_{10}q_0)$
y	$\frac{1}{2}mq_0$	$-\frac{1}{\sqrt{\pi}}\left(3mq_0 - \sqrt{y_m} + \frac{4}{\sqrt{\pi}}m\sqrt{y_m}a_{10}\right)(a_{11} - m\sqrt{y_m}a_{10}q_0) - m\sqrt{y_m}a_{10}q_1$
$y^{\frac{3}{2}}$	$\frac{1}{3}mq_1 + \frac{4}{3\pi}m(a_{11} - m\sqrt{y_m}a_{10}q_0)$	$\frac{16}{3\pi}\Big\{-a_{22} - \frac{1}{\sqrt{\pi}}my_ma_{10}a_{21} - m\sqrt{y_m}a_{21}q_0 - \frac{2}{\sqrt{\pi}}m^2y_m^{\frac{3}{2}}a_{10}a_{20}q_0 - m^2y_ma_{20}q_0^2 - \frac{\sqrt{\pi}}{2}m^2\sqrt{y_m}a_{10}q_0q_1 - \frac{1}{2}m^2y_ma_{10}^2q_1 + \frac{\sqrt{\pi}}{8}my_ma_{10}q_1 - \frac{1}{2}m\sqrt{_m}a_{10}q_2 + \Big[-\frac{2}{\pi}m^2y_ma_{10}^2 - \frac{16}{3\pi^{\frac{3}{2}}}ma_{11} + \frac{1}{\sqrt{\pi}}my_ma_{10} + \frac{1}{16}y_m + \frac{1}{\sqrt{\pi}}\left(\frac{16}{3\pi} - \frac{5}{2}\right)m^2\sqrt{y_m}a_{10}q_0 + \frac{1}{2}m^2q_0^2 + \frac{7}{8}m\sqrt{y_m}q_0 - \frac{4}{3\sqrt{\pi}}mq_1\Big](a_{11} - m\sqrt{y_m}a_{10}q_0)\Big\}$
y^2	$\frac{1}{4}mq_2 + \frac{1}{4}m\Big\{m\sqrt{y_m}a_{10}q_1 + \frac{4}{\sqrt{\pi}}my_ma_{20}q_0 + \frac{2}{\sqrt{\pi}}\sqrt{y_m}a_{21} + \frac{1}{\sqrt{\pi}}\left(mq_0 - \sqrt{y_m} + \frac{4}{\sqrt{\pi}}m\sqrt{y_m}a_{10}\right)\left(a_{11} - m\sqrt{y_m}a_{10}q_0\right)\Big\}$	

and then solves equations (3.5) for the coefficients c_i, $d_i (i = 1, 2\ldots)$ [8]. The resulting system is given in the Appendix, the first four terms for constant applied heat flux are given in Table 1 and the first three terms for arbitrary applied heat flux in Table 2.

A numerical solution of the melt problem valid for all times will now be obtained for a solid cylinder in which the basic equations of the embedding techniques, equations (3.5), are used. Numerical integration of these equations is not straight-forward for the two principal reasons: a Taylor expansion for the dependent variables cannot be employed, and great loss of accuracy is incurred unless special precautions are taken.

The Taylor expansion cannot be written since

$$Q'(0) = \xi(0) = \dot{\xi}(0) = 0$$

and

$$\frac{dQ'(0)}{dy} = \frac{d^nQ'(0)}{dy^n} = \frac{d^n\xi(0)}{dy^n} \to \infty, n = 2, 3\ldots$$

when there is no jump in applied flux at the instant of melt initiation; furthermore, when there is a jump, then $Q'(0)$ and

$$\frac{d^n Q'(0)}{dy^n} \to \infty, \quad n = 1, 2 \ldots.$$

The method employed here eliminates this difficulty by approximating the intervals on the left side of the interface boundary conditions, equations (3.5), as explained below, by means of the previously derived analytical starting solution in the region in which it is valid [12].

Let $y^* > 0$ be a time, sufficiently short, so that the short-time solution holds in good approximation within $0 < y < y^*$, and let

$$\bar{Q}(y) = Q'(y) \qquad 0 < y < y^* \tag{3.8}$$

where $Q'(y)$ is the short-time solution for the fictitious heat flux and y^* is dependent upon the number of terms taken in approximating this function. Then

$$\int_0^y Q'(y - y_1)\frac{\partial G}{\partial y_1}\,dy_1 \cong \int_0^{y-y^*} Q'(y - y_1) \times \frac{\partial G}{\partial y_1}\,dy_1 + \int_{y-y^*}^{y} \bar{Q}(y - y_1)\frac{\partial G}{\partial y_1}\,dy_1 \tag{3.9}$$

where $G(X, y)$ is the appropriate one of the functions T_0 or $\partial T_0/\partial x$ depending upon whether equation (3.5a) or (3.5b) is being considered.

The interface boundary conditions will now be considered for the times $y = y^* + n\delta$, $n = 0, 1, \ldots, \delta > 0$. Before giving the general forms of equations (3.5), for the values of $n > 1$, the special cases of equation (3.5b) for $y = y^*$ and $y = y^* + \delta$ (i.e. $n = 0, 1$) will be studied.

With $y = y^*$ in equation (3.5b) one obtains

$$\frac{d\xi(y^*)}{dy} \cong \frac{m}{2}\left\{\frac{Q(y^*)}{Q_0} - \frac{1}{RT_m}\frac{\partial T^*}{\partial X} + \int^{y^*} \frac{\bar{Q}(y - y_1)}{Q_0}\frac{\partial^2 T_0}{\partial X \partial y_1}\,dy_1\right\}$$
$$X = 1 - 2\sqrt{y_m}\,\xi(y^*). \tag{3.10}$$

The only unevaluated term on the right side of this equation, the term containing the integral, can be determined by direct integration, which yields a very complicated expression involving the Dawson integral [10], or by an approximate numerical scheme. The latter approach was used and the trapezoidal approximation was employed. In equation (3.10) $\xi(y^*)$ is obtained from the short-time solution, and the value of $\xi(t^* + \delta)$ is determined from

$$\xi(y + \delta) \cong \xi(y) + \delta\dot{\xi}(y), \tag{3.11}$$

evaluated at $y = y^*$.

With $y = y^* + \delta$ in equation (3.5b) one obtains

$$\frac{d\xi}{dy}(y^* + \delta) \cong \frac{m}{2}\left\{\frac{Q^*(y^* + \delta)}{Q_0} - \frac{1}{RT_m}\frac{\partial T^*}{\partial X} + \int_0^{\delta} \frac{Q'(y - y_1)}{Q_0}\frac{\partial^2 T_0}{\partial X \partial y_1}\,dy_1 + \int_{\delta}^{y^*+\delta} \frac{\bar{Q}(y - y_1)}{Q_0}\frac{\partial^2 T_0}{\partial X \partial y_1}\,dy_1\right\}$$
$$X = 1 - 2\sqrt{(y_m)}\,\xi(y^* + \delta). \tag{3.12}$$

The last integral on the right is evaluated by the trapezoidal rule. In the first integral the trapezoidal approximation is not sufficiently accurate since $\partial^2 T_0/\partial X \partial y$, varies greatly in the interval in question; however, $Q'(y)$ does not vary significantly and can therefore be approximated by

$$Q'(y^* + \delta - y_1) = Q'(y^*) + \frac{Q'(y^*)}{\delta}(\delta - y_1). \tag{3.13}$$

After substitution of this in the first integral and integration by parts equation (3.12) becomes

$$\frac{d\xi}{dy}(y^* + \delta) \cong \frac{m}{2}\left\{\frac{Q^*(y^* + \delta)}{Q_0} - \frac{1}{RT_m}\frac{\partial T^*}{\partial X} + \frac{Q'(y^*)}{Q_0}\left[\frac{\partial T_0}{\partial X}(X, \delta)\right.\right.$$

$$+ \frac{1}{\delta}\int_0^{\delta} T_0 \,\mathrm{d}y_1 \Bigg] + \int_{\delta}^{y^*+\delta} \frac{\bar{Q}(y - y_1)}{Q_0} \times \frac{\partial^2 T_0}{\partial X \partial y_1}\mathrm{d}y_1 \Bigg\}$$

$$X = 1 - 2\sqrt{(y_m)}\xi(y^* + \delta). \quad (3.14)$$

Letting $y = y^* + \delta$ in equation (3.11) one obtains $\xi(y^* + 2\delta)$.

The general formulation for $y > y^* + \delta$ can now be considered. The first integral on the right side of equation (3.9) is broken into n intervals of duration δ and the trapezoidal rule is applied to the last $(n - 1)$ intervals. In the first interval, $0 < y_1 < \delta$, for small $\xi(y)$, the function $G(X, y)$ again varies considerably, as in the case of the evaluation of $\dot{\xi}(y^* + \delta)$, while $Q'(y)$ does not. Consequently, in this interval, $Q'(y)$ is approximated by

$$Q'(y - y_1) \cong Q'(y - \delta) + \frac{\delta - y_1}{\delta}[Q'(y - \delta) - Q'(y - 2\delta)]. \quad (3.15)$$

After integration by parts equation (3.9) becomes

$$\int^{y} Q'(y - y_1)\frac{\partial G}{\partial y_1}\mathrm{d}y_1 \cong Q'(y - \delta)\Bigg\{G(X, \delta) + \frac{1}{\delta}\int_0^{\delta} G(X,y_1)\,\mathrm{d}y_1\Bigg\} - \frac{Q'(y - 2\delta)}{\delta}\int_0^{\delta} G\,\mathrm{d}y_1$$
$$+ \frac{\delta}{2}\Bigg\{Q'(y - \delta)\frac{\partial G}{\partial y_1}(X,\delta) + Q'(y^*) \times \frac{\partial G}{\partial y_1}(X,y - y^*)\Bigg\} + \delta\sum_{i=2}^{n} Q'(y - i\delta)$$
$$\times \frac{\partial G}{\partial y_1}(X,i\delta) + \int_{y - y^*}^{y} \bar{Q}(y - y_1) \times \frac{\partial G}{\partial y_1}(X,y_1)\,\mathrm{d}y_1. \quad (3.16)$$

As before the last integral on the right is evaluated by the trapezoidal rule. Substitute this result into equations (3.5) to get the two interface conditions as:

$$\frac{Q'(y - \delta)}{Q_0}\Bigg\{T_0(X,\delta) + \frac{1}{\delta}\int_0^{\delta} T_0\,\mathrm{d}y_1 + \frac{\delta}{2}T_0(X,\delta)\Bigg\} \cong \frac{1}{R}\Bigg\{1 - \frac{T^*}{T_m}(X,y)\Bigg\}$$
$$+ \frac{Q'(y - 2\delta)}{Q_0}\frac{1}{\delta}\int_0^{\delta} T_0\,\mathrm{d}y_1 + \frac{\delta}{2}\frac{Q'(y^*)}{Q_0} \times \frac{\partial T_0}{\partial y_1}(X,y - y^*) - \delta\sum_{i=2}^{n}\frac{Q'(y - i\delta)}{Q_0}$$
$$\times \frac{\partial T_0}{\partial y_1}(X,i\delta) - \int_{y-y^*}^{y}\frac{\bar{Q}(y - y_1)}{Q_0}\frac{\partial T_0}{\partial y_1}\mathrm{d}y_1$$

$$X = 1 - 2\sqrt{(y_m)}\xi(y) \quad (3.17a)$$

$$\frac{2}{m}\frac{\mathrm{d}\xi}{\mathrm{d}y} \cong \frac{Q(y)}{Q_0} - \frac{1}{RT_m}\frac{\partial T^*}{Q_0} - \frac{Q'(y - \delta)}{Q_0} \times \Bigg\{\frac{\partial T_0}{\partial X}(X,\delta) + \frac{1}{R}\int_0^{\delta}\frac{\partial T_0}{\partial X}\mathrm{d}y_1 + \frac{\delta}{2}\frac{\partial^2 T_0}{\partial X \partial y_1}(X,\delta)\Bigg\} - \frac{Q'(y - 2\delta)}{Q_0}$$
$$\times \frac{1}{\delta}\int_0^{\delta}\frac{\partial T_0}{\partial X}\mathrm{d}y_1 - \frac{\delta}{2}\frac{Q'(y^*)}{Q_0}\frac{\partial^2 T_0}{\partial X \partial y}(X,y - y^*)$$
$$+ \delta\sum_{i=2}^{n}\frac{Q'(y - i\delta)}{Q_0}\frac{\partial^2 T_0}{\partial X \partial y_1}(X,i\delta) + \int_{y-y^*}^{y}\frac{\bar{Q}(y - y_1)}{Q_0}\frac{\partial^2 T_0}{\partial X \partial y_1}\mathrm{d}y,$$

$$X = 1 - 2\sqrt{(y_m)}\xi(y). \quad (3.17b)$$

With $y = y^* + n\delta$ and $\xi(y^* + n\delta)$ known, $Q'[y^* + (n-1)\delta]$ and $\dot{\xi}(y^* + n\delta)$ are determined from equations (3.17) and $\xi[y^* + (n+1)\delta]$ is determined from equation (3.11). Then n is increased and the process is repeated.

This procedure led to sufficiently accurate results for values of the dimensionless melt depth ξ not larger than about 0·20; however, for larger values the results became inaccurate. This occurred because an attempt was being made, in the above procedures, to adjust a flux on the surface [i.e. $Q'(y)$] to give the correct temperature at an interior point; the latter is however little influenced by relatively large changes in this flux; in other words, it is characteristic of the diffusion problem that a small inaccuracy in the heat flux applied to the surface of a body will, for short times, have little effect on the temperature of points distant from the surface. The reverse is true of the converse problem which we are solving, that is, a small inaccuracy in the calculation of the temperature may result in a large error in the calculation of the surface heat flux. More specifically, in our problem calculations are made according to the formula

$$\frac{T^*}{T_m} + \frac{T'}{T_m} + \frac{T_1}{T_m} = 1, \tag{3.18}$$

where T^* is the temperature at the melt surface due to the analytic continuation of the pre-melt heat flux and the temperature T' and T_1 are caused by the fictitious heat input Q' as follows:

$$\frac{T'}{T_m} = \sum_{i=2}^{n} c_i Q'(y - i\delta)$$

$$\frac{T_1}{T_m} = c_1 Q'(y - \delta). \tag{3.19}$$

The procedure consists in calculating T_1/T_m from equation (3.18) and then solving for $Q'(y - \delta)$ by determining c_1. This latter calculation is difficult to perform accurately for the reasons discussed in the preceding paragraph and further, because T_1/T_m is the small difference of two numbers of almost equal magnitude, $T^*/T_m + T'/T_m \cong 1$, and unity. It was found that as $\xi(y)$ approached a value for which T_1/T_m became inaccurate, $\xi \cong 0{\cdot}2$, up to six figures were being lost in the calculation of $(1 - T^*/T_m - T'/T_m)$.

Two approaches were used to remedy this situation.* In the first the unknown values of the function $Q'(y)$ were extrapolated in the manner presently to be shown and equations (3.17b) and (3.11) were used to solve for $\xi(y)$ until melting stopped. The extrapolated values of $Q'(y)$ were determined from

$$Q'(y) \cong Q'(\bar{y}) + \frac{\mathrm{d}Q'(\bar{y})}{\mathrm{d}y}(y - \bar{y}) + \frac{1}{21}\frac{\mathrm{d}^2Q'(\bar{y})}{\mathrm{d}y^2}(y - \bar{y})^2, \tag{3.20}$$

where $\bar{y}$ is the time of the last reliably calculated value of $Q'(y)$ and the coefficients $\dot{Q}'(\bar{y})$ and $\ddot{Q}'(\bar{y})$ were determined by the second order backwards difference formulas:

$$\frac{\mathrm{d}Q'(\bar{y})}{\mathrm{d}y} \cong \frac{1}{2\delta}\left\{3Q'(\bar{y}) - 4Q'(\bar{y} - \delta) + Q'(\bar{y} - 2\delta)\right\}, \tag{3.21a}$$

$$\frac{\mathrm{d}^2Q'(\bar{y})}{\mathrm{d}y^2} \cong \frac{1}{\delta^2}\{2Q'(\bar{y}) - 5Q'(\bar{y} - \delta) + 4Q'(\bar{y} - 2\delta) - Q'(\bar{y} - 3\delta)\}. \tag{3.21b}$$

This procedure is quite limiting, however, even if more terms are taken in equation (3.20), since it cannot be used where it would be unreasonable to expect an expansion of the form of equation (3.20), i.e. if the applied heat flux is not sufficiently smooth, or where a check on the approximation is not available. In such cases an alternate procedure can be used to solve the system of equations. Two equations of (3.17a) for successive time $y = y^* + n\delta$ and $y = y^* + (n+1)\delta$, are subtracted and a predictor–corrector method for $Q'(y)$ is employed. Subtracting the two

* A third approach [5] would employ increasing time intervals as time increases, but the accuracy of the calculation would thereby be diminished.

equations helps improve the accuracy of the calculations since the difference form does not change T_1/T_m as given in equation (3.19) while the other terms in equation (3.19) are made significantly smaller. Equation (3.19) thus becomes

$$\frac{\Delta T^*}{T_m} + \frac{\Delta T'}{T_m} = \frac{T_1}{T_m} \tag{3.22}$$

This decreases the effect of the second source of error described above, while the predictor–corrector method similar to that of [7] decreases the effect of the first source of error. Thus T_1/T_m is determined by using an extrapolation formula to evaluate $Q'(y-\delta)$. The effect due to $Q'(y-2\delta)$, which is found to be significantly larger than that due to T_1/T_m, is then extracted from T'/T_m, i.e.

$$\frac{\Delta T'}{T_m} = \Delta c_2 Q'(y-2\delta) \sum_{i=3}^{n} \Delta c_1 Q'(y-i\delta) \tag{3.23}$$

and $Q'(y-2\delta)$ is treated as an unknown. With $\xi(y)$ known, a value of $Q'(y-\delta)$ is predicted, the difference equation form of equation (3.17a) is solved for $Q'(y-2\delta)$ and $\dot{\xi}(y)$ is determined in the normal manner. The process is then repeated for the next incremented value of y to obtain the corrected value of $Q'(y-\delta)$ and the value of $\dot{\xi}(y+\delta)$. This method effectively extends the range over which $Q'(y)$ can be determined.

4. NUMERICAL RESULTS

The first approach, as described in the preceding section, was employed in solving various problems with constant applied heat flux. A check was provided by equation (2.10), which in nondimensional form is:

$$y_f = 2\sqrt{(y_m)} \int_0^{y_f} \xi(y_1)\,\mathrm{d}y_1 + \frac{1}{2\sqrt{(y_m)}mQ_1}\left[1 + \frac{2m}{\sqrt{(\pi)}}\right] - 1, \tag{4.1}$$

where y_f is the total time to melt and Q_1 is the actual heat flux divided by the reference Q_0. Table 3, which presents y_f for various values of the material parameter m and the parameter y_m which is a measure of the magnitude of the applied flux or the melt time, shows that the first approach is valid over a wide range of parameters. Lines 5 and 6 of the table compare y_f for the same value of the parameters m and y_m with different approximations for equation (3.20), i.e. in line 5, $\dot{Q}'(\bar{y})$ was set equal to zero and in line 6 it was calculated from equation (3.21b). It can be seen that the per cent difference between the predicted numerical results is decreased to 1·26 per cent in the quadratic approximation from 4.81 per cent in the linear approximation.

Table 3. Comparison of numerical and predicted melt times

	m	y_m	mR	y_f (numerical)	y_f (pred.)	% difference
1	0·5	0·01	4·431	28·26	28·34	0·286
2	0·5	0·1	1·401	8·65	8·58	0·875
3	0·5	0·4	0·701	4·97	5·00	0·600
4	5	0·1	14·01	2·14	2·21	3·27
5	10	0·4	14·01	0·603 (linear)	0·632	4·81
6	10	0·4	14·01	0·636 (quadratic)	0·664	1·26

In Fig. 3 curves of melt depth vs. time for various values of the parameter $mR = \sqrt{(\pi)}m/2\sqrt{(y_m)}$ are presented. Although only a few values of mR are used, the curves indicate that, as mR is increased, the normalized plots of s against time depend essentially on the single parameter mR. A similar result was obtained by Citron in [11] for the case of a slab.

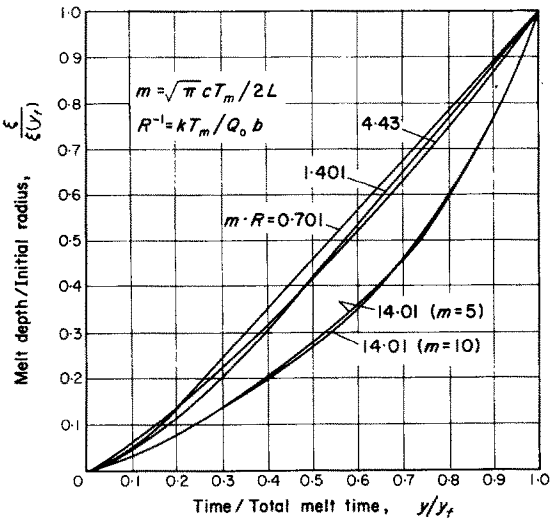

FIG. 3. Melt depth vs. time.

The second method was used on a case of the variable applied heat flux shown in Fig. 4. This flux, presented in [11], is typical of fluxes due to atmospheric heating of reentry vehicles. As shown in the figure a reasonable approximation for this flux, during the period $t < 19$ s, is

$$Q(t) = 11{\cdot}776\,(e^{0{\cdot}152t} - 1) \qquad t < 19 \text{ s}. \tag{4.2}$$

This results in the following temperature distribution before melting:

$$T(r, t) = 23{\cdot}552 \frac{\kappa}{b^2} \frac{b}{kT_m} \left\{ \frac{1}{0{\cdot}152}(e^{0{\cdot}152t} - 1) - t + \sum_{s=1}^{\infty} \left[\!\left[\frac{e^{0{\cdot}152t}}{0{\cdot}152 + \frac{\alpha_s^2 \kappa}{b^2}} \left[1 - \exp\left\{ -\left(0{\cdot}152 + \frac{\alpha_s^2\kappa}{b^2}\right)t \right\} \right] \frac{J_0(\alpha_s r/b)}{J_0(\alpha_s)} + \frac{b^2}{\kappa}\left(\exp - \frac{\alpha_s^2 \kappa b}{b^2} - 1\right) \frac{J_0(\alpha_s r/b)}{J_0(\alpha_s)} \right]\!\right] \right\}. \tag{4.3}$$

A solid iron cylinder of radius 2 cm with physical properties $\kappa = 0{\cdot}0986$ cm^2/s, $T_m = 1500$°C, $L = 64$ cal/g, $\rho = 7{\cdot}5$ g/cm^3, $c = 0{\cdot}13$ cal/g°C and $m = 2{\cdot}7$ was considered.

The melt time was determined from the transcendental equation which results from setting $T = T_m$, $r = b$ and $t = t_m$ in equation (4.3). This results in the value $t_m = 17{\cdot}45$ s. Since $t_m < 19$ s, equations (4.2) and (4.3) may be used as the analytic continuations of the pre-melt heat flux and temperature distribution, respectively. The results showing melt depth are given in Fig. 4.

It may be concluded that the fictitious body technique is a workable method for obtaining numerical solutions of melting and solidification problems. It requires a minimum of calculations, as compared to a finite difference scheme using a space-time grid, since only quantities on the melt interface are calculated, and thus ordinary, rather than partial, integro-differential equations are used. Of course, great care must be exercised in order that sufficient accuracy is maintained; the check provided by the time of total melt, equation (4.1), is valuable in this connection.

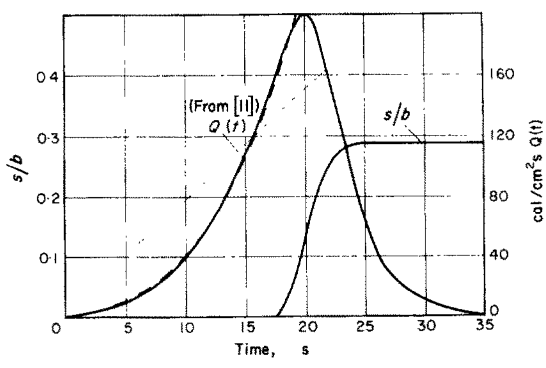

FIG. 4. Melt depth vs. time for a cylindrical body subjected to atmospheric heating during re-entry.

5. DISCUSSION OF SHORT-TIME SOLUTION

Conditions for the starting of melting

Melting may or may not start when the surface of the cylinder first reaches the melt temperature depending upon the history of heating and the applied flux at $t = t_m$. Conditions on the applied flux can be determined from the fact that the melt depth must initially become positive for melting to start. From the solution for $\xi(y)$, Table 2, one sees that $q_0 > 0$ is sufficient to guarantee that melting starts at t_m. If $q_0 = 0$ then the condition becomes $q_1 + 4a_{11}/\pi > 0$, in which case the applied flux may even be decreasing, though of course it must remain positive. Should $q_0 = q_1 + 4a_{11}/\pi = 0$ then the condition becomes $[q_2 - \dot{Q}(0^-)] - \sqrt{(y_m a_{11}/\sqrt{\pi}} > 0$ and so forth. For the special case of a heat flux analytic about $y = 0$, the second condition is seen to be satisfied, since $q_1 = 0$ and $a_{11} > 0$.

Comparison between two problems with instantaneous melt removal

Two solutions (denoted by the superscripts 1 and 2 respectively) will now be considered for the problem with instantaneous melt removal under two heat inputs Q^1 and Q^2, i.e. solutions to equations (2.1)–(2.8) with $a(t) = s(t)$. It will be shown that, if $[rQ^1(r, t)|_{r=b-s^1}] = [rQ^2(r, t)|_{r=b-s^2}]$ in $0 < t < t' \geqslant t_m$ and $[rQ^1(r, t)|_{r=b-s^1}] > [rQ^2(r, t)|_{r=b-s^2}]$ in $t' < t$, so that $Q^1 = Q^2$ in $0 < t < t'$ and $Q^1 \neq Q^2$ for $t > t'$, then $s^1(t) > s^2(t)$ for $t > t'$.

The statement is proved in two parts: in the first it is shown that s^1 is initially greater than s^2, while in the second it is shown that the assumption that s^2 is greater than s^1 results in a contradiction at later times. The first part of the proof follows from the overall heat balance, equation (2.9), which written for both problems and subtracted becomes, after subtracting,

$$0 < \int_{t^1}^{t^1+\delta} [(b - s^1)\, Q^1(b - s^1_, t) - (b - s^2)\, Q^2(b - s^2, t)]\, dt$$

$$= \int_{b_0}^{b-s^2} \rho_S c_S u_S r\, dr - \int^{b-s^1} [\rho_S c_S (T_m - T_S^1) + \rho_S l_S]\, r\, dr. \qquad (5.1)$$

Assume in some small interval $t' < t < t' + \delta$, $\delta > 0$ that $s^2 > s^1$; then applying the lemma derived in [13] and noting that $u_s = T_S^1 - T_S^2 \leqslant 0$ yields

$$0 < -\int_{b-s^2}^{b-s^1} [\rho_S c_S (T_m - T_S^1) + \rho_S l_S]\, r\, dr, \qquad (5.2)$$

which results in a contradiction since the integrand is positive; hence $s^1 > s^2$ in $t' < t < t' + \delta$.

It is possible that a time $t'' > t' + \delta$ exists for which $s^2(t'') = s^1(t'')$. It can however, be shown that such a time cannot exist provided $(rQ|_{b-s}) > (rQ^2|_{b-s})$. The proof is not given since it follows an analogous proof for a slab [12], except that $rQ|_{b-s}$ replaces $Q(t)$ and $\frac{d}{dt}[rQ|_{b-s(t)}]$ replaces dQ/dt.

An example in which the conditions of the above comparison theorem holds is immediately given by the short-time solution of Table 2, since when $q_0^1 > q_0^2$ it is clear that $s^1 > s^2$. For this example not only is $(rQ^1|_{b-s^1}) > (rQ^2|_{b-s^2})$, but $Q^1|_{b-s^1} > Q^2|_{b-s^2}$ also holds. It is of course possible to have $Q^1|_{b-s^1} > Q^2|_{b-s^2}$ and $s^1 > s^2$ with $(rQ^1|_{b-s^1})$ not always larger than $(rQ^2|_{b-s^2})$ provided that sufficiently large times are considered. An example of this

$$q_0^1 - q_0^2 = \Delta q_0 > 0, \quad q_1^1 = q_1^2 = 0,$$
$$q_2^1 - q_2^2 = -\Delta q_2, \quad q_0^1 \gg 0, \quad q_0^2 \gg 0, \qquad (5.3a)$$

in which case in the interval $0 < y < \Delta q_0/\Delta q_2$, $\xi^1 > \xi^2$ and

$$X\frac{Q^2}{Q_0}\bigg|_{X=1-2\sqrt{y_m}\xi^2} - X\frac{Q^1}{Q_0}\bigg|_{X=1-2\sqrt{y_m}\xi^1} \cong -(\Delta q_0 - \Delta q_2 y) + m\Delta q_0\sqrt{(y_m)}\,\frac{Q(0^-)}{Q_0} + q_0^1 + q_0^2 \quad y. \qquad (5.3b)$$

Thus it is seen for q_0^1, q_0^2 sufficiently large there is a period within the interval $0 < y < \Delta q_0/\Delta q_2$ in which $(rQ^1|_{b-s^1}) < (rQ^2|_{b-s^2})$.

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APPENDIX

The following is the set of algebraic equations obtained from equations (3.5) with equations (3.7) substituted

$$\frac{\pi}{4}c_1 - 2\sqrt{(y_m)}a_{10}d_2 = -a_{11}, \qquad \text{(A.1a)}$$

$$\frac{2}{3}c_2 - \sqrt{(\pi)}c_1d_2 + \frac{\sqrt{(\pi y_m)}}{6}c_1 + 2\sqrt{(y_m)}a_{10}d_3 = 0, \qquad \text{(A.1b)}$$

$$\frac{3\pi}{16}c_3 + \frac{\sqrt{(\pi y_m)}}{8}c_2 + \frac{3\pi}{64}y_mc_1 - \sqrt{(\pi)}c_1d_3 + 2\pi c_1d_2^2 + 2\sqrt{(y_m)}a_{10}d_4 + 2\sqrt{(y_m)}a_2d_2 + 4y_ma_{20}d_2^2 = -a_{22} \qquad \text{(A.1c)}$$

$$\frac{8}{15}c_4 + \frac{\sqrt{(\pi y_m)}}{10}c_3 + \frac{y_m}{10}c_2 + \frac{\sqrt{(\pi)}}{20}y_m^{\frac{3}{2}}c_1 - \sqrt{(\pi)}c_1d_4 - \sqrt{(\pi)}c_2d_3 - \sqrt{(\pi)}c_3d_2 + 4\pi c_1d_2d_3 + 2c_2d_2^2 - \sqrt{(\pi y_m)}\,c_1d_2^2 - 2\sqrt{(y_m)}a_{10}d_5 + 2\sqrt{(y_m)}a_{21}d_3 + 8y_ma_{20}d_2d_3 = 0, \qquad \text{(A.1d)}$$

$$\frac{2}{m}d_2 = q_0, \qquad \text{(A.2a)}$$

$$\frac{3}{m}d_3 + c_1 = q_1, \qquad \text{(A.2b)}$$

$$\frac{4}{m}d_4 + c_2 - \sqrt{(\pi)}c_1d_2 - \frac{8}{\sqrt{(\pi)}}y_ma_{20}d_2 = q_2 + a_{21}\bigg/\frac{Q_0b}{kT_m}, \qquad \text{(A.2c)}$$

$$\frac{5}{m}d_5 + c_3 - \sqrt{(\pi)}c_1d_3 - \frac{4}{\sqrt{(\pi)}}c_2d_2 + c_1d_2^2 - \frac{8}{\sqrt{(\pi)}}y_ma_{20}d_3 = q_3, \qquad \text{(A.2d)}$$

$$\frac{6}{m}d_6 + c_4 - \sqrt{(\pi)}c_1d_4 - \frac{4}{\sqrt{(\pi)}}c_2d_3 - \frac{3\sqrt{(\pi)}}{2}c_3d_2 - \frac{3\sqrt{(\pi)}}{8}y_mc_1d_2 + 2c_1d_2d_3 + \frac{4}{3}c_2d_2^2 - \frac{8}{\sqrt{(\pi)}}y_ma_{20}d_4 - \frac{24}{\sqrt{(\pi)}}y_m^{\frac{3}{2}}a_{30}d_2^2 = q_4 + a_{32}\bigg/\frac{Q_0b}{kT_m}. \qquad \text{(A.2e)}$$

Equations A.1 are obtained from the interface temperature condition and equations A.2 from the interface heat balance. Notice they can be solved for any given set of q_i by direct elimination. Table 2 gives a few terms of the solution for arbitrary heat flux.

FUSION OU SOLIDIFICATION A SYMETRIE DE REVOLUTION DE CYLINDRES CIRCULAIRES

Résumé—Une solution analytique pour les temps courts et les solutions numériques pour un instant quelconque sont obtenues pour la fusion d'un cylindre avec des flux de chaleur arbitraires imposés sur son rayon extérieur. Les solutions sont obtenues par la technique d'encastrement; la précision des solutions est discutée et quelques résultats numériques sont présentés.

ACHSYMMETRISCHES SCHMELZEN ODER VERFESTIGEN VON KREISFÖRMIGEN ZYLINDERN.

Zusammenfassung—Eine analytische Lösung für kurze Zeiten und numerische Lösungen für beliebige Zeiten wurden für beliebige Zeiten wurden für einen Zylinder ermittelt bei beliebiger Wärmezufuhr am Umfang. Die Lösungen wurden mit Hilfe der "Einbautechnik" erhalten; ihre Genauigkeit wird diskutiert und an numerischen Ergebnissen gezeigt.

ОСЕСИММЕТРИЧНОЕ ПЛАВЛЕНИЕ ИЛИ ЗАТВЕРДЕВАНИЕ КРУГОВЫХ ЦИЛИНДРОВ

Аннотация—Получены аналитическое для малых времен и численные решения для любых моментов времени о плавлении цилиндров при произвольной подаче тепла на его внешнюю боковую поверхность. Решения получены с помощью методики вложения. Обсуждается точность решений, и приводятся некоторые численные результаты.